Adaptive Optimal Basis Tomography for Qubits

Syed Muhammad Kazim, Junaid ur Rehman, Kyesan Lee, and Hyundong Shin
Department of Electronics and Information Convergence Engineering, Kyung Hee University, Yongin-si, 17104 Korea
Email: hshin@khu.ac.kr

***Abstract*—We present a two-level adaptive quantum state tomography protocol. On the first level, we utilize the particle distribution at each iteration to optimize the measurement setting of the next. On the second level, we use measurement outcomes to periodically reduce the set of measurement settings. We demonstrate the usefulness of this method by reporting infidelity and its standard deviation.**

I. Introduction

Characterization of quantum states has become an essential task with the emergence of quantum technologies. It has important implications in research areas such as quantum computing, optics and communication [1]–[3]. Complete or full quantum state tomography (QST) is the field which formally deals with the task of statistically identifying an arbitrary quantum state, given the availability of a sufficient amount of identically prepared states [4], [5]. Other popular branches of tomography include characterization of quantum logic gates and Hamiltonian studied under Gate Set tomography and Hamiltonian tomography, respectively [6], [7].

Given an ensemble of identically prepared qubits of ρ, QST employs a series of measurements, and uses the outcomes of each measurement to reconstruct another quantum state $\hat{\rho}$, which estimates ρ. Statistical quantification also requires the description of errors in our estimate [8], [9]. As the number of qubits measured increase, the errors in approximation decrease since the relative frequency approaches true frequency in probability. However, in the absence of infinitely many qubits (which is the case in real life), we require a method that reconstructs a physically valid quantum state which reduces errors to below a fixed threshold (of a chosen figure of merit) utilizing as few resources as possible.

A valid qubit is described by a density matrix which is positive semi-definite, Hermitian with unit trace. Bayesian inference methods are used to ensure validity of the reconstructed state and provide its complete statistical quantification [8]. A generic implementation of Bayesian methods use three constituents namely prior, likelihood and evidence (or denominator). These elements produce a posterior distribution. Bayesian mean estimation (BME) methods use the mean of the posterior as an estimate. A more powerful and computationally tractable implementation of Bayesian schemes is known as the particle filter (PF) [10]. In the quantum version of this method, prospective states are distributed according to a prior. After every measurement, we update the likelihood of these particles [11].

PF methods can be made adaptive seamlessly [12], [13]. In this paper, we utilize the BME of each posterior to find a new measurement setting. Moreover, we keep track of the outputs in each configuration, and successively reduce this set by eliminating elements with high entropy outcomes. In the end, our set configurations consists of just one element, which we demonstrate encapsulates the entire information required to reconstruct the density matrix of the quantum state.

This paper is structured as follows. We expound the basics of PF, required to understand its application to state tomography in section II. Section III explains the two-layered adaptive nature of our protocol. In section IV, we apply our understanding to simulate quantum state tomography of qubit, provide results, and analyze our posterior. We conclude in section V.

II. Primer on Particle Filter

In the case of qubits, an arbitrary ρ can be represented in term of the Bloch vector $\boldsymbol{r}$ [5]

$$\rho = \frac{1}{2}\left(\mathbb{I} + \boldsymbol{r}\cdot\boldsymbol{\sigma}\right), \tag{1}$$

where $\mathbb{I}$ is a 2×2 identity matrix, $\boldsymbol{r} = (r_x, r_y, r_z) \in \mathbb{R}^3$, and $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z)$ is a vector of Pauli operators. To ensure validity of the state, $\|\boldsymbol{r}\|_2 \leq 1$ where $\|\cdot\|_2$ is the Euclidean norm.

For QST, we initially measure ρ in configurations $\boldsymbol{\sigma}_i$ for $i \in \{x, y, z\}$. This set of configurations varies in adaptive protocols as we shall see in section III. We observe outcome $|\ell\rangle$ for $\ell \in \{+1_i, -1_i\}$ for each i with probability $P\,|\ell\rangle = \operatorname{tr}\left(|\ell\rangle\langle\ell|\,\rho\right)$. Then the task of QST is to best approximate these probabilities to estimate ρ.

To apply QST using PF, we initialize particles $\{\gamma_k\}$ for $k \in \{1, \cdots, k\}$ in the Bloch space based on prior distribution. After measurement of N_0 qubits in σ_i, and observing n_ℓ outcomes of ℓ, we evaluate the likelihood function

$$\mathcal{L}\left(\gamma_k\right) = N_0! \prod_{\ell} \frac{\operatorname{tr}\left(|\ell\rangle\langle\ell|\,\gamma_k\right)^{n_\ell}}{n_\ell!}. \tag{2}$$

The distribution of the normalized likelihoods of particles known as its weights w_k forms the posterior distribution for this iteration.

III. Two-layered Adaptive Method

Measurement of any ρ in its own diagonal basis provides the best possible scaling of $O\left(\frac{1}{N}\right)$ [14]. In real life cases, when the diagonal basis is not available, we can instead use the diagonal bases of BME estimates of each iteration. In the first layer of adaptivity, we initialize our measurement set with

Pauli matrices. After each measurement, we apply a unitary on our original set based on the BME of the posterior distribution. This is analogous to rotating the measurement set such that the diagonal basis of one of the elements of our set (on average) more closely approximates the diagonal basis of ρ [13].

As these iterations continue, we observe that measurement outcomes of two of the operators are $n_\ell \approx \frac{N_0}{2}$. In information theoretic terms, these configurations maximize entropy. For QST, if we visualize the Bloch sphere in terms of rotated axis of $\tilde{\boldsymbol{\sigma}}$, then $\hat{\rho}$ lies close to only one axis. Using this insight, we understand that measurements on the operators which maximize entropy in this setting will not provide us with any meaningful information, and hence can be eliminated from our measurement set. In this manner, we successively reduce our measurement set till we only have one element. This element encapsulates the entire qubit information required for QST in this rotated Bloch space. This forms the second layer of our adaptive protocol.

IV. Results

In this section, we demonstrate the performance of our method detailed previously. We simulate 200 random two dimensional mixed states for 1000 iterations of $N_0 = 50$ shots each. For We report infidelity $\mathcal{I}$ between the estimated states $\hat{\rho}$ and true states ρ defined as

$$\mathcal{I}(\rho, \hat{\rho}) = 1 - \sqrt{\mathrm{tr}\left(\sqrt{\rho}\hat{\rho}\sqrt{\rho}\right)}. \tag{3}$$

The infidelity captures the idea of closeness between ρ and $\hat{\rho}$ such that $\mathcal{I}(\rho, \hat{\rho}) = 0$ if and only if $\rho = \hat{\rho}$. Figure 1 demonstrates the scaling of infidelity of our protocol for against iterations.

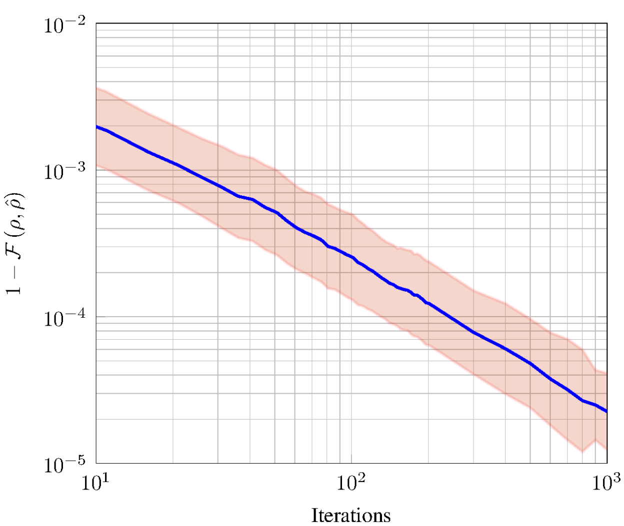

Fig. 1. Infidelity for the proposed protocol averaged over 200 mixed states with $N_0 = 50$ shots per iteration. The red shaded region indicates 16 % and 84% quantiles over all measurements.

V. Conclusions

In this work, we formulated QST as a PF problem. We have highlighted fundamentals of PF required to apply Bayesian Inference in the domain of state tomography. Moreover, we have we have provided a novel adaptive scheme that optimizes the measurement configurations and successively reduces the cardinality of the measurement set. The implications of this method are more pronounced for higher dimensional qudits where an informationally complete measurement set can be very large. This formulation, therefore, not only improves scaling but also makes higher dimensional problems computationally tractable.

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